SELF-CONSISTENT SOLUTIONS OF SOME PROBLEMS
OF HEAT AND MASS TRANSFER IN A SATURATED
POROUS MEDIUM

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Self-consistent solutions of systems of heat conduction and filtration equations in regions separated by mobile boundaries are considered.

The heat-conduction and filtration equations describing the processes of heat and mass transfer in saturated porous media are encountered in a whole series of problems, including: the drying and purification of porous bodies [1]; the destruction of solids, accompanied by physicochemical conversions and the subsequent transfer of individual components within the porous material formed [2, 4]; thermal methods of intensifying the extraction of useful minerals — high-viscosity petroleum, bitumin, sulfur, gaseous hydrates [5-7], etc. The mathematical formulation of these problems involves complicated versions of the Stefan problem.

In the present work, self-consistent solutions of one class of such problems are obtained in a more general formulation. The case considered is that in which the pressure and temperature at a mobile boundary are unknown, but the functional thermodynamic phase-equilibrium dependences relating them is known. In contrast to certain of the works cited above, more precise conditions of material and thermal balance at the mobile boundary are given.

Suppose that a saturated porous medium occupies the right-hand half space $x > 0$, and that at the surface $x = 0$ heat is supplied to the medium. With time, the temperature of the medium increases, and the material to be extracted, initially present in the pores in the solid state, gradually melts. The melting surface $l(t)$ separating the liquid and solid phases moves to the right, and the liquid phase filters to the left; this transfer may be produced by pressure reduction at the boundary $x = 0$ using pumping out of the melt or else as a result of increase in pressure inside the medium by heating [2, 3].

Heat loss due to heat transfer with the surrounding medium is neglected. Use is also made of the fact that the time for the liquid-phase temperature to equalize with the temperature of the porous-medium mass is much less than the total duration of the heating process [7].

The temperature distribution in the first region may be described, in the case of constant thermophysical parameters, by the linearized equation of convective heat transfer [5]

$$\lambda_{\text{I}} \frac{\partial^2 T_{\text{I}}}{\partial x^2} - c_1 \rho_s v \frac{\partial T_{\text{I}}}{\partial x} = \alpha_{\text{I}} \frac{\partial T_{\text{I}}}{\partial t},\tag{1}$$

where $\lambda_1 = \lambda_0 (1-m) + \lambda_1 m$ and $\alpha_1 = \rho_1 c_1 m + \rho_0 c_0 (1-m)$ are the mean thermal conductivity and bulk specific heat of the first region; ρ_s is the mean liquid-phase density; and

$$v = -\frac{k}{\mu} \frac{\partial p}{\partial x}.\tag{2}$$

Here and below, the subscript 0 refers to the body of pores; 1 to the liquid phase; 2 to the solid phase; I to the region $0 < x < l(t)$; and II to the region $l(t) < x < \infty$.

In region II no liquid phase is present, and therefore the temperature distribution is described by the equation

$$a_{II} \frac{\partial^2 T_{II}}{\partial x^2} = \frac{\partial T_{II}}{\partial t} \quad l(t) < x < \infty, \tag{3}$$

where

$$a_{II} = \frac{\lambda_{II}}{\alpha_{II}}, \quad \alpha_{II} = (1 - m)\rho_0 c_0 + m\rho_2 c_2, \quad \lambda_{II} = (1 - m)\lambda_0 + m\lambda_2.$$

The following boundary conditions are taken in solving Eqs. (1) and (3)

$$T_I(0, t) = T_G, \quad T_{II}(x, 0) = T_{II}(\infty, t) = T_0, \tag{4}$$

$$T_I(l, t) = T_{II}(l, t). \tag{5}$$

At the surface $l(t)$, the heat-balance condition is satisfied

$$-\lambda_1 \frac{\partial T_I}{\partial x} + c_1\rho_s T(l, t) v(l, t) + \lambda_{II} \frac{\partial T_{II}}{\partial x} = [\rho_2 L + c_1\rho_s T(l, t)] m \frac{dl}{dt}. \tag{6}$$

The pressure distribution in region I is described by the non-steady-filtration equation

$$\varkappa \frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t}. \tag{7}$$

For this equation the following conditions are satisfied:

$$p(0, t) = p_G, \tag{8}$$

$$v(l, t) = -m \frac{\rho_2 - \rho_s}{\rho_s} \frac{dl}{dt}. \tag{9}$$

The last condition expresses the mass balance at the limit of solid-phase melting.

In the general case, the pressure and temperature at the surface $l(t)$ are unknown, but are related by the equation of the phase-equilibrium curve

$$T(l, t) = F[p(l, t)] = T_p. \tag{10}$$

As a rule, the dependence $T = F(p)$ is determined experimentally in each specific case.

The problem in Eqs. (1)-(10) is closed, and admits of self-consistent solution.

The solution of Eq. (7) is sought in the form

$$p(z) = A_1 \operatorname{erf} z + A_2, \tag{11}$$

where $\operatorname{erf} z$ is the probability integral; $z = x/2\sqrt{\varkappa t}$ is a self-consistent variable.

It follows from Eq. (8) that $A_2 = p_G$. Substituting Eq. (11) into Eq. (9) gives

$$A_1 = \frac{m\mu}{k} \frac{\rho_2 - \rho_s}{\rho_s} \exp\left(\frac{l^2}{4\varkappa t}\right) \sqrt{\pi \varkappa t} \frac{dl}{dt} = \text{const}. \tag{12}$$

Hence, it follows that

$$\sqrt{\pi \varkappa t} \frac{dl}{dt} = \text{const}, \quad \frac{l^2(t)}{4\varkappa t} = \text{const} = \beta^2, \tag{13}$$

where β is an arbitrary constant.

Then Eq. (12) takes the form

$$A_1 = \sqrt{\pi} \frac{m\mu}{k} \frac{\rho_2 - \rho_s}{\rho_s} \beta\varkappa \exp(\beta^2).$$

The final result obtained for Eq. (11) is

$$p = p_G + \frac{m\mu}{k} \frac{\rho_2 - \rho_s}{\rho_s} \sqrt{\pi} \beta \varkappa \exp(\beta^2) \operatorname{erf} z. \tag{14}$$

It follows from Eqs. (1) and (14) that

$$v = -\frac{\theta_0}{\sqrt{t}} \exp(-z^2), \quad \theta_0 = m \frac{\rho_2 - \rho_s}{\rho_s} \beta \sqrt{\varkappa} \exp(\beta^2). \tag{15}$$

Substituting Eq. (14) into Eq. (1) gives

$$a_I \frac{\partial^2 T_I}{\partial x^2} + \frac{\theta}{\sqrt{t}} \exp\left(-\frac{x^2}{4\varkappa t}\right) \frac{\partial T_I}{\partial x} = \frac{\partial T_I}{\partial t}, \tag{16}$$

where

$$a_I = \frac{\lambda_I}{\alpha_I}; \quad \theta = \frac{c_1 \rho_s}{\alpha_I} \theta_0.$$

Assuming that T_I and T_{II} are, respectively, functions of the self-consistent variables

$$z_1 = \frac{x}{2\sqrt{a_I t}}, \quad z_2 = \frac{x}{2\sqrt{a_{II} t}},$$

Eqs. (16) and (3) are written in the form of ordinary differential equations

$$\frac{d^2 T_I}{dz_1^2} + 2\left[z_1 + \frac{\theta}{\sqrt{a_I}} \exp\left(-\frac{z_1^2}{v^2}\right)\right] \frac{dT_I}{dz_1} = 0, \quad v = \sqrt{\frac{\varkappa}{a_I}}, \tag{17}$$

$$\frac{d^2 T_{II}}{dz_2^2} + 2 z_2 \frac{dT_{II}}{dz_2} = 0. \tag{18}$$

Integrating these equations, and determining the constants of integration from Eqs. (4), (5), and (10), it is found that

$$T_I = T_G - (T_G - T_p) \frac{\Phi(z_1)}{\Phi(\gamma_1)}, \tag{19}$$

$$T_{II} = T_0 + (T_p - T_0) \frac{\operatorname{erfc} z_2}{\operatorname{erfc} \gamma_2}, \tag{20}$$

$$\gamma_1 = \frac{l(t)}{2\sqrt{a_I t}} = \text{const}, \quad \gamma_2 = \frac{l(t)}{2\sqrt{a_{II} t}} = \text{const}, \tag{21}$$

$$\operatorname{erfc} \xi = 1 - \operatorname{erf} \xi, \quad \Phi(\xi) = \int_0^{\xi} \exp\left[-\sigma^2 + \theta v \sqrt{\frac{\pi}{a_I}} \operatorname{erf} \frac{\sigma}{v_1}\right] d\sigma_1.$$

The pressure at the melting surface is found from Eq. (14)

$$p(l(t), t) = p_G + \frac{m\mu}{k} \frac{\rho_2 - \rho_s}{\rho_s} \sqrt{\pi} \beta \varkappa \exp(\beta^2) \operatorname{erf} \beta. \tag{22}$$

Then, in accordance with Eq. (10), the unknown temperature T_p in Eqs. (19) and (20) is determined with the known dependence

$$T_p = F[p(l(t), t)].$$

The law of melting-surface motion may be determined from the first relation in Eq. (21)

$$l(t) = 2\gamma_1 \sqrt{a_I t}. \tag{23}$$

Thus, Eqs. (14), (19), (20), and (23) comprise the solution of the problem, which will be completely solved if the constants β, γ_1, and γ_2 are determined.

It follows from Eqs. (13), (21), and (23) that

$$\beta = \frac{\gamma_i}{\nu}, \quad \gamma_2 = \gamma_i \sqrt{\frac{a_1}{a_{11}}}.$$

Substitution of the values of $v(l, t)$, T_I, T_{II}, and $l(t)$ into Eq. (18) gives, taking the last two relations into account, a transcendental equation for γ_1

$$\frac{\lambda_I (T_G - T_p)}{2\sqrt{a_I}} \frac{\exp\left[-\gamma_1^2 - q\gamma_i \exp\left(\frac{\gamma_i}{\nu}\right)^2 \operatorname{erf}\frac{\gamma_i}{\nu}\right]}{\Phi(\gamma_i)}$$

$$- \frac{\lambda_{II}(T_p - T_0)}{\sqrt{\pi a_{II}}} \frac{\exp\left(-\gamma_1^2 \frac{a_I}{a_{II}}\right)}{\operatorname{erfc}\left(\gamma_i \sqrt{\frac{a_I}{a_{II}}}\right)} = (L + c_i T_p) m\rho_2 \gamma_i \sqrt{a_i}, \tag{24}$$

$$q = \sqrt{\pi}\nu m \frac{\rho_2 - \rho_s}{\alpha_i} c_i.$$

The value of $T_p = T_p(\gamma_i)$ is determined from Eqs. (10) and (23).

The expressions obtained may be simplified in the case of quasisteady liquid-phase filtration. In fact, in saturated porous media as a rule, $\varkappa \gg a_I$, i.e., the pressure perturbation propagates much more rapidly than the temperature perturbation [7]. The quantity γ_1, characterizing the velocity of melting-surface motion, is usually of the order of unity or less. Then

$$\nu = \sqrt{\frac{\varkappa}{a_1}} \gg 1, \quad \beta = \frac{\gamma_i}{\nu} \ll 1.$$

Setting

$$\exp \beta^2 = 1, \quad \operatorname{erf}\frac{x}{2\sqrt{\varkappa t}} \leqslant \operatorname{erf}\beta \frac{x}{l(t)} \approx \frac{2}{\sqrt{\pi}}\beta\frac{x}{l(t)}$$

in Eq. (14) yields an expression for the pressure

$$p = p_G + \frac{2m\mu_i}{k} \frac{\rho_2 - \rho_s}{\rho_s} \beta^2 \varkappa \frac{x}{l(t)}. \tag{25}$$

It is simple to establish that this expression is the solution of Eq. (7) for $\varkappa = \infty$ and satisfies the boundary conditions in Eqs. (8) and (9).

The constant β, as before, is determined from Eq. (13). The filtration rate is

$$v = -\frac{\theta_0'}{\sqrt{t}}, \quad \theta_0' = m\beta\sqrt{\varkappa} \frac{\rho_2 - \rho_s}{\rho_s},$$

and, accordingly, there are no exponential expressions in Eqs. (16) and (17), and θ is replaced by the quantity

$$\theta' = \frac{c_i \rho_s}{\alpha_i} \theta_0'.$$

The solution of Eq. (17) takes the form

$$T_1 = T_G - (T_G - T_p) \frac{\operatorname{erf}(z_i + \bar{\theta}) - \operatorname{erf}\bar{\theta}}{\operatorname{erf}(\gamma_i + \bar{\theta}) - \operatorname{erf}\bar{\theta}}, \quad \bar{\theta} = \frac{\theta'}{\sqrt{a_1}}. \tag{26}$$

Note that this expression may also be obtained directly from Eq. (19) if the function $\Phi(\xi)$ is simplified by the means indicated above, and the integral is evaluated.

Thus, in the case of quasisteady filtration, the pressure and temperature in the first region are determined from Eqs. (25) and (26), while Eqs. (20) and (23) remain unchanged. To determine γ_1, Eqs. (6), (20), (23), (25), and (26) yield the equation

$$\frac{\lambda_1 (T_G - T_p)}{\sqrt{\pi a_I}} \cdot \frac{\exp(-\gamma_1 - q'\gamma_1)^2}{\operatorname{erf}(\gamma_1 + q'\gamma_1) - \operatorname{erf} q'\gamma_1} - \frac{\lambda_{II}(T_p - T_0)}{\sqrt{\pi a_{II}}}$$

$$\times \frac{\exp\left(-\gamma_1^2 \dfrac{a_I}{a_{II}}\right)}{\operatorname{erfc} \gamma_1 \sqrt{\dfrac{a_I}{a_{II}}}} = (L + c_1 T_p) m\rho_2 \gamma_1 \sqrt{a_I}, \quad q' = \frac{m(\rho_2 - \rho_s)c_1}{\alpha_I}. \tag{27}$$

Consider, as an example, the melting of a bitumen mass in a porous medium using a heater. The following parameters were adopted in the calculations for the thermophysical quantities characteristic of petroleum deposits: $\rho_0 = 2500$ kg/m^3; $\rho_1 = 800$ kg/m^3; $\rho_2 = 950$ kg/m^3; $c_0 = 750$ J/kg·deg; $c_1 = 2000$ J/kg·deg; $c_2 = 2000$ J/kg/deg; $\lambda_0 = 1.38$ W/m·deg; $\lambda_1 = 0.17$ W/m·deg; $\lambda_2 = 0.5$ W/m·deg; $L = 167 \cdot 10^3$ J/kg; $a_I = 0.625 \cdot 10^{-6}$ m^2/sec; $a_{II} = 0.708 \cdot 10^{-6}$ m^2/sec; $T_0 = 30°C$; $T_p = 60°C$.

The values of γ_1 calculated from Eq. (27) for a temperature drop $T_G - T_p$ in the first region of 40, 90, and 180°C are 0.425, 0.875, and 1.125, respectively.

Knowing γ_1, the expressions given above may be used to calculate such characteristics of the given process as the temperature and pressure fields, the liquid-phase filtration rate, and the coordinate of the melting surface as a function of the time. For example, the melting depth $l(t)$ of the bitumen mass determined from Eq. (23) after 100 days with the above temperature drops are found to be 2.19, 4.50, and 5.78 m, respectively.

NOTATION

x	is the coordinate;
t	is the time;
T	is the temperature;
T_G	is the temperature at the coordinate origin;
T_p	is the phase-transition temperature;
T_0	is the initial temperature;
λ	is the thermal conductivity;
c	is the specific heat;
ρ	is the density;
v	is the filtration rate;
k	is the permeability;
m	is the porosity;
μ	is the viscosity;
$\varkappa$	is the piezoconduction coefficient;
a	is the thermal diffusivity;
$l(t)$	is the phase-transition surface;
p	is the pressure;
p_G	is the pressure at coordinate origin;
p_p	is the pressure at phase-transition surface;
z, z_1, z_2	are the self-consistent variables;
A_1 and A_2	are the constants of integration;
β, γ_1, γ_2	are the constants;
erf	is the probability-integral symbol.

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USE OF WATSON OPERATIONS IN THE SOLUTION
OF SOME PROBLEMS IN THE THEORY OF
THERMAL CONDUCTIVITY

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The Watson transform method is used to obtain analytical solutions of certain nonstationary problems in the theory of thermal conductivity for variable regions with a specified law of boundary motion.

We will consider the following problem: we must find a solution of the thermal conductivity equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

in the region $l_1 + bt \leq x \leq l_2 + bt$, $l_2 - l_1 = l > 0$, $-\infty < t < \infty$, which satisfies the initial condition

$$u\big|_{t=-\infty} = 0 \tag{2}$$

and the boundary conditions

$$\frac{\partial u}{\partial x} + \alpha u \bigg|_{x=l_1+bt} = h_1(t), \tag{3}$$

$$u\big|_{x=l_2+bt} = h_2(t). \tag{4}$$

Here a, b, l_1, l_2, α are constant parameters. We will seek the solution of Eqs. (1)–(4) in the form of the sum of thermal potentials of a simple and twin layer [1, 2]

$$u(x, t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_1(s)}{\sqrt{t-s}} e^{-\frac{(x-l_1-bs)^2}{4a^2(t-s)}} ds + \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \rho_2(s) \frac{(x-l_2-bs)}{(t-s)^{3/2}} e^{-\frac{(x-l_2-bs)^2}{4a^2(t-s)}} ds, \tag{5}$$

where $\rho(t)$ and $\rho_2(t)$ are unknown functions, to be defined from boundary conditions (3), (4); initial condition (2) is satisfied automatically. To define $\rho_1(t)$ and $\rho_2(t)$ we obtain a system of Voltaire integral equations of the second sort

$$h_i(t) = (-1)^i \frac{\rho_i(t)}{2} + \sum_{j=1}^{2} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) \rho_j(s) ds \quad (i = 1, 2), \tag{6}$$

where

$$K_{11}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{b^2}{4a^2} x\right),$$

$$K_{12}(x) = \left(\frac{a^2 + lb - a^2\alpha l}{a^2 x^{3/2}} - \frac{l^2}{2a^2 x^{5/2}} - \frac{b^2 + 2a^2 b}{2a^2 \sqrt{x}}\right) \exp\left(-\frac{b^2 x}{4a^2} + \frac{lb}{4a^2} - \frac{l^2}{4a^2 x}\right),$$

$$K_{21}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{b^2 x}{4a^2} - \frac{lb}{4a^2} - \frac{l^2}{4a^2 x}\right),$$